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SCIENCE DIRECT
Journal of Functional Analysis 200 (2003) 71–100
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The explicit duality correspondence of $(Sp(p,q), O^*(2n))$

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Received 22 March 2002; accepted 2 August 2002

Abstract

We investigate the type I dual pairs over the quaternion algebra $\mathbb{H}$, namely the family of dual pairs $(Sp(p,q), O^*(2n))$. We give a complete and explicit description of duality correspondence for $p+q \leqslant n$ as well as some of the cases for $p+q > n$, in terms of the Langlands parameters.

1. Introduction

One of the most serious difficulties in the determination of the unitary dual of any simple Lie group is to understand the unitary representations which are "singular" in a well defined sense. In [9,14] a large set of singular unitary representations were constructed for classical groups via the method of reductive dual pairs. Suppose G, G' is a pair of reductive groups which are centralizers of each other in some ambient symplectic group, and that they are in the so-called stable range (roughly, this means G' is at most half the size of G). Then there is an injection from the

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[1] Supported in part by NNSFC Grant No. 19928103, RGC-CERG Grants HKUST6169/99P, HKUST6126/00P, and the Cheung Kong Scholars Programme at Zhejiang University.
[2] Supported in part by the NUS-MOE Grants R-146-000-003-112 and R-146-000-017-112, respectively.

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PII: S0022-1236(02)00079-4

unitary dual of G' to that of G. Keeping G fixed and varying G', we obtain families of singular unitary representations of G which are parametrized by the unitary duals of various G' (of much smaller size).

Unfortunately, the unitary representations of G thus obtained were described by restriction to certain maximal parabolic subgroups. The construction provides no a priori information about the Langlands parameters of these representations. What [14] does prove, however, is the fact that the injections agree with the local theta correspondences. Thus it is of considerable interest to describe, in terms of the Langlands–Vogan parameters, the local theta correspondence for reductive dual pairs in the stable range.

For applications to automorphic forms and other reasons, it is of great interest to understand the theta correspondence in general. Indeed, there have been numerous papers on the subject. In terms of explicit descriptions, the papers [1,2,16,17] are among the most important. However, complete descriptions were obtained (for the type of dual pairs considered in these papers) for more or less the 'equal rank' cases only (with the exception of [1] for complex groups). In particular, very little is known for real reductive dual pairs in the stable range.

In this paper we investigate the type I dual pairs over the quaternion algebra $\mathbb{H}$, namely the family of dual pairs $(Sp(p,q), O^*(2n))$. We shall give a complete and explicit description of duality correspondence for $p+q \leqslant n$ as well as some of the cases for $p+q>n$, in terms of the Langlands parameters. We note that in this case equal rank means $p+q=n$ or $n-1$, while "stable range" means $p+q \leqslant n/2$.

Our approach is very natural and will be briefly outlined now. Let $\hat{G}_{\mathrm{d}}$ denote the equivalence classes of discrete series representations of G. Let $\hat{G}$ denote the admissible dual. From [15] we can read off a bijection

$$\widehat{O^*(2n)}_{\mathrm{d}} \leftrightarrow \bigcup_{p+q=n,n-1} \widehat{Sp(p,q)}_{\mathrm{d}}. \tag{1.1}$$

Now an arbitrary irreducible admissible representation can be obtained as a Langlands subquotient of some standard representation induced from relative discrete series. By applying the induction principle of Kudla and Moeglin (as in [1,2,16,17]) and going through a careful analysis of lowest K-types, we obtain a bijection

$$\widehat{O^*(2n)} \leftrightarrow \bigcup_{p+q=n,n-1} \widehat{Sp(p,q)}, \tag{1.2}$$

where $\hat{G}$ denotes the admissible dual of G.

Our approach and results up to this point are quite close to those of several others mentioned above. The key and simple observation here is that a similar approach is in fact capable of handling a range far beyond the equal rank case, at least for the family of dual pairs under consideration. This goes as follows.

Suppose that $p+q \leqslant n-2$. We write $n = m+2k$ with $m = p+q$ or $p+q+1$. Starting from any $\pi \in \widehat{Sp(p,q)}$ we first obtain the theta lift $\theta_m(\pi)$ of π to $O^*(2m)$ (see Notation 3.31) via the Theorem 1.3. Consider the parabolic subgroup of $O^*(2n)$ with Levi subgroup $O^*(2m) \times GL(1,\mathbb{H})^k$. It turns out that $\theta_n(\pi) \in \widehat{O^*(2n)}$ is a subquotient of the representation induced from $\theta_m(\pi) \otimes \chi$, where χ is a one-dimensional representation of the (non-abelian) group $GL(1,\mathbb{H})^k$. Again sophisticated versions of the induction principle and careful lowest K-type analysis allow us to identify the Langlands parameters of $\theta_n(\pi)$. Our main result can be stated as

Theorem. *For any integers p, q, n with $p+q \leqslant n$, the theta correspondence gives rise to an injection*

$$\widehat{Sp(p,q)} \hookrightarrow \widehat{O^*(2n)}, \tag{1.3}$$

which is explicitly described in terms of the Langlands parameters.

The explicit description of this injection is given in Theorem 5.8. When $p+q > n$, a similar technique can be applied to obtain the theta lift of $\pi' \in \widehat{O^*(2n)}$ to $Sp(p,q)$ if (p,q) falls in the same Witt tower determined by the signature of π'. See Theorem 5.23.

As an immediate corollary, we obtain families of irreducible small unitary representations of $O^*(2n)$ (with their Langlands parameters) from (the Langlands parameters of) known unitary representations of $Sp(p,q)$ with $p+q \leqslant n/2$. The latter set includes the trivial representation, tempered representations, unitary representations with non-zero cohomology. Most of the unitary representations of $O^*(2n)$ obtained this way were not known previously. In particular, for $n \geqslant 8$ this gives rise to a complete classification of irreducible unitary representations of $O^*(2n)$ with rank $\leqslant 3$. Here the notion of *rank* is as defined by Howe [8]. For more discussions, see Section 5.3.

Finally, for the sake of completeness, we also work out the duality correspondence for $(GL(m,\mathbb{H}), GL(n,\mathbb{H}))$, the type II dual pairs over $\mathbb{H}$.

2. The groups and their representations

2.1. The groups

For any positive integer n, we denote the $n \times n$ identity matrix I_n. If p and q are non-negative integers, let

$$Sp(p,q) = \{g \in Sp(2(p+q),\mathbb{C}) | {}^t\bar{g} K_{p,q} g = K_{p,q}\},$$

where $K_{p,q}$ is the diagonal matrix $diag(I_p, -I_q, I_p, -I_q)$. The group $O^*(2n) = SO^*(2n)$ is the real form of $SO(2n, \mathbb{C})$ realized as

$$O^*(2n) = \left\{ g \in O(2n, \mathbb{C}) | {}^t\bar{g} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \right\}$$

(see [5]).

Using the identification $\mathbb{C}^2 \to \mathbb{H}$ given by $(a, b) \mapsto a + jb$ (see [11]), we can think of $Sp(p, q)$ as the isometry group of the hermitian form $(,)$ on $V = \mathbb{H}^{p+q}$ over the quaternion algebra $\mathbb{H}$ given by

$$(v, w) = \sum_{i=1}^{p} \bar{v}_i w_i - \sum_{i=p+1}^{p+q} \bar{v}_i w_i$$

for $v, w \in V$, $v = (v_1, \dots, v_{p+q})$, and $w = (w_1, \dots, w_{p+q})$. Similarly, the group $O^*(2n)$ may be thought of as the isometry group of the skew-hermitian form $(,)'$ on $V' = \mathbb{H}^n$ given by $(u, z)' = \sum_{i=1}^{n} u_i j \bar{z}_i$.

2.2. The representations

To describe the admissible representations of $Sp(p, q)$ and $O^*(2n)$, we use the parametrization of [23]. We denote Lie algebras by gothic letters with subscript 0, and omit the subscript to denote complexified Lie algebras.

We realize $\mathbb{H}$ as

$$\mathfrak{u}^*(2) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \middle| z_1, z_2 \in \mathbb{C} \right\}$$

and the group $GL(1, \mathbb{H}) = \mathbb{H}^\times$ as $U^*(2) = \mathfrak{u}^*(2) \cap GL(2, \mathbb{C})$. Notice that $GL(1, \mathbb{H}) \cong SU(2) \times \mathbb{R} \cong Sp(1) \times \mathbb{R}$, so that its representations may be parametrized by pairs (μ, v) with μ a positive integer and v a complex number. The complexified Lie algebra of $GL(1, \mathbb{H})$ is $\mathfrak{gl}(2, \mathbb{C})$. The representation $\sigma(\mu, v)$ given by (μ, v) corresponds to $S^{\mu-1}(\mathbb{C}^2) \otimes det(\cdot)^{\frac{v}{2}}$ with $det(\cdot)$ the reduced norm of $\mathbb{H}$ (or equivalently, $det(h)$ is the determinant of h as an element of $U^*(2)$) and $S^k(\mathbb{C}^2)$ the irreducible $(k+1)$-dimensional representation of $SU(2)$. The infinitesimal character of $\sigma(\mu, v)$ is $(\frac{v}{2} + \frac{\mu}{2}, \frac{v}{2} - \frac{\mu}{2})$.

Now let $G = Sp(p, q)$ or $O^*(2n)$, K a maximal compact subgroup of G, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition, and $\mathfrak{t} \subset \mathfrak{k}$ a Cartan subalgebra.

If $G = Sp(p, q)$, we choose a basis $\{e_1, \dots, e_p, f_1, \dots, f_q\}$ of $i\mathfrak{t}^*$ so that the compact roots are

$$\Delta_c = \{\pm 2e_i, \pm 2f_i, \pm(e_i \pm e_j), \pm(f_i \pm f_j)\},$$

and the non-compact roots are $\Delta_n = \{\pm(e_i \pm f_j)\}$. We choose a system of positive compact roots Δ_c^+ so that one half the sum of the positive compact roots is

$$\rho_c = (p, p-1, \dots, 2, 1; q, q-1, \dots, 2, 1).$$

If $G = O^*(2n)$, we choose a basis $\{e_1, e_2, \dots, e_n\}$ of $i\mathfrak{t}^*$ so that

$$\Delta_c = \{\pm(e_i - e_j) | i<j\},$$

and $\Delta_n = \{\pm(e_i + e_j)|i<j\}$. Now choose Δ_c^+ so that

$$\rho_c = (\tfrac{n-1}{2}, \tfrac{n-3}{2}, \dots, \tfrac{-n+1}{2}).$$

Then the discrete series of G is parametrized by Harish–Chandra parameters λ as follows:

If $G = Sp(p,q)$,

$$\lambda = (a_1, a_2, \dots, a_p; b_1, \dots, b_q) \tag{2.1}$$

with $a_i, b_j \in \mathbb{Z}$, $a_1 > a_2 > \cdots > a_p \geqslant 1$, $b_1 > b_2 > \cdots > b_q \geqslant 1$, and $a_i \neq b_j$ for all i, j.

If $G = O^*(2n)$,

$$\lambda = (a_1, a_2, \dots, a_n) \tag{2.2}$$

with $a_1 > a_2 > \cdots > a_n$, $a_i + a_j \neq 0$, and $a_i \in \mathbb{Z}$.

Limits of discrete series of G are given by pairs (λ, Ψ), where $\lambda \in i\mathfrak{t}_0^*$ and $\Psi \subset \Delta(\mathfrak{g} : \mathfrak{t})$ is a system of positive roots such that λ is dominant for Ψ. Moreover, $\Delta_c^+ \subset \Psi$, and if α is a simple root in Ψ and $\langle \lambda, \alpha \rangle = 0$ then α is non-compact. (This is condition F-1 of [23].) The unique lowest K-type of the limit of discrete series $\pi = \pi(\lambda, \Psi)$ is then $\Lambda = \lambda + \rho_n - \rho_c$, where ρ_n is one half the sum of the non-compact roots in Ψ. (We identify K-types with their highest weights.)

If $G = Sp(p,q)$, then λ is of the form

$$\lambda = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}; \overbrace{a_1, \dots, a_1}^{n_1}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}), \tag{2.3}$$

where $a_i \in \mathbb{Z}$, $a_1 > \cdots > a_k > 0$ and $|m_j - n_j| \leqslant 1$.

If $G = O^*(2n)$, then

$$\lambda = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \tag{2.4}$$

with the a_i as in (2.3); or with one zero between the last a_k and the first $-a_k$.

Using the parametrization of [23], each irreducible admissible representation of G may be given by a θ-stable Cartan subgroup $H = TA$ and a final limit character $\gamma = (\Psi, \Gamma, \bar{\gamma})$ as follows.

For this section only, let $n = p + q$ if $G = Sp(p, q)$. Denote the split rank of G by

$$r_G = \begin{cases} \min\{p, q\} & \text{if } G = Sp(p, q), \\ [\frac{n}{2}] & \text{if } G = O^*(2n). \end{cases} \tag{2.5}$$

Up to conjugation by K, the θ-stable Cartan subgroups of G are given by $H_r = T_r A_r$ with $0 \leqslant r \leqslant r_G$ with $T_r \cong U(1)^{n-r}$ and $A_r \cong \mathbb{R}^r$. Notice that $T = T_0 \cong U(1)^n$. The centralizer $M_r A_r$ of A_r is a Levi subgroup of G with relative discrete series. We have

$$M_r A_r = G_r \times GL(1, \mathbb{H})^r, \tag{2.6}$$

$$H_r = U(1)^{n-2r} \times (\mathbb{C}^\times)^r, \tag{2.7}$$

$$(\mathfrak{h}_r)_0 = \mathfrak{u}(1)^{n-2r} \times \mathfrak{u}(1)^r \times \mathbb{R}^r. \tag{2.8}$$

Here $G_r = Sp(p - r, q - r)$ if $G = Sp(p, q)$, and $O^*(2(n - 2r))$ if $G = O^*(2n)$. The limit character γ corresponds to inducing data as follows. Let $\rho = \rho(\lambda, \Psi)$ be a limit of discrete series representation of G_r, $\sigma = \otimes_{j=1}^r S^{\mu_j - 1}(\mathbb{C}^2)$ a representation of $SU(2)^r$, and $\chi_v = \prod_{j=1}^r \chi_{v_j}$ the character of $\mathbb{R}^r$ given by $\chi_{v_i}(r) = e^{v_i r}$, $v = (v_1, \dots, v_r) \in \mathbb{C}^r$. Let

$$\begin{aligned} \bar{\gamma}|_{\mathfrak{u}(1)^{n-2r}} &= \lambda, \\ \bar{\gamma}|_{\mathfrak{u}(1)^r} &= (\mu_1, \dots, \mu_r), \\ \bar{\gamma}|_{\mathbb{R}^r} &= v. \end{aligned} \tag{2.9}$$

The non-parity condition F-2 of [23] amounts to the requirement that μ_i is odd if $v_i = 0$. Γ is the character of H_r determined by the condition $d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$ (notice that H_r is connected). Choose $P_r = M_r A_r N_r$ so that $Re\{\langle \alpha, \bar{\gamma} \rangle\} \leqslant 0$ $\forall \alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$. Then $X(\gamma) = Ind_{P_r}^G(\rho \otimes \sigma \otimes \chi_v)$ has a unique irreducible submodule. Here, as everywhere in this paper, we use normalized induction.

Definition 2.10. We write $\pi(r, \lambda, \Psi, \mu, v)$ to denote the (infinitesimal equivalence class) of the unique irreducible G submodule of $X(\gamma)$ as above, where $r \leqslant r_G$, (λ, Ψ) indexes a limit of discrete series of G_r, and $\mu \in (\mathbb{Z}_{\geqslant 1})^r$, $v \in \mathbb{C}^r$.

This parametrizes the irreducible admissible representations of G up to conjugation by K. It is easy to check that $\pi(r, \lambda, \Psi, \mu, v) \simeq \pi(r', \lambda', \Psi', \mu', v') \Leftrightarrow r = r'$, $\lambda = \lambda'$, $\Psi = \Psi'$, and (μ', v') may be obtained from (μ, v) by simultaneous permutation of the coordinates, and by replacing some of the v_i by $-v_i$.

Recall that $\pi(r, \lambda, \Psi, \mu, v)$ may also be obtained as a submodule of a standard module $X(\gamma_d)$ which is induced from a discrete series representation. This module is then the sum of all modules $X(\gamma')$ for γ' a final limit character associated to a fixed

Cartan subgroup $H_{r'}$ and a fixed parameter $\bar{\gamma}'$. We obtain one such module for each root system Ψ' which can be associated to the parameter λ (as in (2.3) and (2.4)), subject to condition F-1 and with $\Delta_c^+ \subset \Psi$. We obtain $X(\gamma_d)$ from $X(\gamma)$ by changing the inducing data as follows. Let $r_d = r + \sum_{i=1}^{k} \min\{m_i, n_i\}$. Then λ_d is obtained from λ by removing, for each $i \leqslant k$, $\min\{m_i, n_i\}$ a_i's on each side of the semicolon if $G = Sp(p,q)$, and $\min\{m_i, n_i\}$ a_i's and $-a_i$'s each if $G = O^*(2n)$. Since λ_d is non-singular, Ψ_d is then uniquely determined. To obtain μ_d, we concatenate μ and

$$(\underbrace{a_1, \ldots, a_1}_{\min\{m_1, n_1\}}, \ldots, \underbrace{a_k, \ldots, a_k}_{\min\{m_k, n_k\}})$$

and $v_d = (v_1, \ldots, v_r, 0, \ldots, 0)$. Notice that γ_d does not satisfy the non-parity condition F-2.

2.3. Lowest K-types

We compute the lowest K-types of the representations of the previous subsection, using the standard theory of [12] and [22]. If γ is a limit character, write $\bar{\gamma} = (\lambda', v)$ with $\lambda' \in \mathfrak{t}^*$. If $\bar{\gamma}$ is as in (2.9), write λ' as the Δ_c^+-dominant $W(T:G)$ conjugate of

$$\begin{aligned}(&\overbrace{a_1, \ldots, a_1}^{m_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k}, \tfrac{\mu_1}{2}, \ldots, \tfrac{\mu_r}{2};\\ &\overbrace{a_1, \ldots, a_1}^{n_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{n_k}, \tfrac{\mu_1}{2}, \ldots, \tfrac{\mu_r}{2})\end{aligned} \tag{2.11}$$

if $G = Sp(p,q)$, and

$$\begin{aligned}(&\overbrace{a_1, \ldots, a_1}^{m_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k}, \tfrac{\mu_1}{2}, \ldots, \tfrac{\mu_r}{2},\\ &-\tfrac{\mu_r}{2}, \ldots, -\tfrac{\mu_1}{2}, \overbrace{-a_k, \ldots, -a_k}^{n_k}, \ldots, \overbrace{-a_1, \ldots, -a_1}^{n_1}),\end{aligned} \tag{2.12}$$

or with one zero between $\frac{\mu_r}{2}$ and $-\frac{\mu_r}{2}$, if $G = O^*(2n)$.

Let $\mathfrak{q} = \mathfrak{q}(\lambda') = \mathfrak{l} \oplus \mathfrak{u}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by λ' (see [22, Definition 5.2.1]). Then the normalizer L of $\mathfrak{q}$ in G is of the form

$$L = U(p_1, q_1) \times U(p_2, q_2) \times \cdots \times U(p_s, q_s) \tag{2.13}$$

with $|p_i - q_i| \leqslant 1$. The lowest K-types of $X(\gamma)$ and $X(\gamma_d)$ are then of the form $\Lambda = \lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L$ for some fine weights δ_L of $K \cap L$. Here $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are one-half the sums of the roots associated to $\mathfrak{u} \cap \mathfrak{p}$ and $\mathfrak{u} \cap \mathfrak{k}$, respectively, and δ_L is the differential of $det^a \otimes det^{-a}$ on each factor $U(p_i) \times U(q_i)$, with $a \in \{0, \pm\frac{1}{2}\}$, and $a = 0$ if $p_i \neq q_i$ (see [17,22]).

A more explicit calculation: Relabel the coefficients of λ' so that it is in the form of (2.3) or (2.4). Notice that the a_j are now integers or half integers. If a_j is a half integer then $a_j = \frac{\mu_i}{2}$ for some i. In this case we have $m_j = n_j$.

Set

$$R_j = m_1 + \cdots + m_j, \quad S_j = n_1 + \cdots + n_j. \tag{2.14}$$

- $G = Sp(p,q)$:
  We have $R_k = p, S_k = q$, and

$$\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$$
$$= (\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}; \overbrace{c_1, \dots, c_1}^{n_1}, \dots, \overbrace{c_k, \dots, c_k}^{n_k}), \tag{2.15}$$

  where

$$b_j = a_j + R_j - S_j - \tfrac{1}{2}(m_j - n_j + 1) + q - p,$$
$$c_j = a_j - R_j + S_j - \tfrac{1}{2}(n_j - m_j + 1) + p - q. \tag{2.16}$$

- $G = O^*(2n)$:

We have $R_k + S_k = n$ or $n - 1$. Then

$$\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$$
$$= (\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}, \overbrace{-c_k, \dots, -c_k}^{n_k}, \dots, \overbrace{-c_1, \dots, -c_1}^{n_1}) \tag{2.17}$$

if $R_k + S_k = n$, or with $R_k - S_k$ in the middle between b_k and $-c_k$ if $R_k + S_k = n - 1$. Here

$$b_j = a_j + R_j - S_j - \tfrac{1}{2}(m_j - n_j + 1),$$
$$c_j = a_j - R_j + S_j - \tfrac{1}{2}(n_j - m_j + 1). \tag{2.18}$$

Then in both cases, the lowest K-types are obtained by changing (b_j, c_j) to $(b_j + \frac{x}{2}, c_j - \frac{x}{2})$ where $x = 0$ or ± 1, chosen so that the result is an integral weight.

3. The correspondence and K-types

3.1. The space of joint harmonics

The following discussion is in [10]. Let (G, G') be a reductive dual pair in $Sp = Sp(2n, \mathbb{R})$, and let $\mathscr{F}$ be the Fock space of the oscillator representation of the double cover $\widetilde{Sp}$. For any subgroup H of Sp, we denote $\tilde{H}$ the inverse image of H in $\widetilde{Sp}$ by the covering map. Recall that the $\tilde{U}(n)$-finite vectors may be realized as the space of polynomials in n variables, in such a way that the action of $\tilde{U}(n)$, and therefore, that of $\tilde{K}$ and $\tilde{K}'$ preserves the degree. This allows us to associate to each $\tilde{K}$- and $\tilde{K}'$-type occurring in $\mathscr{F}$ a degree, which is the minimal degree of polynomials in the isotypic subspace.

There is a $\tilde{K}\times\tilde{K}'$-invariant subspace $\mathscr{H}$ of $\mathscr{F}$, the space of joint harmonics, with the following properties.

Theorem 3.1 (Howe). *There is a one–one correspondence of $\tilde{K}$- and $\tilde{K}'$-types on $\mathscr{H}$ with the following properties. Suppose π and π' are irreducible admissible representations of $\tilde{G}$ and $\tilde{G}'$ respectively, and $\pi\leftrightarrow\pi'$ in the correspondence for the dual pair (G,G'). Let σ be a $\tilde{K}$-type occurring in π, and suppose that σ is of minimal degree among the $\tilde{K}$-types of π. Then σ occurs in $\mathscr{H}$. Let σ' be the $\tilde{K}'$-type which corresponds to σ in $\mathscr{H}$. Then σ' is a $\tilde{K}'$-type of minimal degree in π'.*

For the dual pairs $(Sp(p,q),O^*(2n))$, the two-fold covers are the trivial (disconnected) ones, so that we can state all the results in terms of the groups themselves, rather than the covering groups.

An explicit description of the correspondence of K-types in the space of joint harmonics for the dual pairs $(Sp(p,q),O^*(2n))$ can be obtained using the known duality correspondence for the case where the first member of the dual pair is compact (see [4]), and the analysis in Section 3 of [10]. The degrees of the K-types in $\mathscr{H}$ may be obtained by considering the see-saw dual pairs

$$\begin{array}{ccc} U(2p,2q) & & O^*(2n) \\ & \times & \\ Sp(p,q) & & U(n) \end{array} \tag{3.2}$$

and the known degrees for K-types in the space of joint harmonics for the dual pairs $(U(p,q),U(r,s))$ (see e.g. [17]).

Fix n,p,q and let $\mathscr{F}(n;p,q)$ be the Fock model for the dual pair $(Sp(p,q),O^*(2n))$. Let σ be an irreducible representation of $Sp(p)\times Sp(q)\subseteq Sp(p,q)$ and σ' an irreducible representation of $U(n)\subseteq O^*(2n)$. Write

$$\begin{cases} \sigma = (a_1,\dots,a_r,0\dots0;b_1,\dots,b_s,0\dots0) & (r\leqslant p,\ s\leqslant q), \\ \sigma' = (a'_1,\dots,a'_{r'},0\dots0,-b'_{s'},\dots,-b'_1) & \\ \qquad +(p-q,\dots,p-q) & (r'+s'\leqslant n), \end{cases} \tag{3.3}$$

where $a_1\geqslant\cdots\geqslant a_r>0,\ b_1\geqslant\cdots\geqslant b_s>0$, and similarly for the a'_i and b'_j.

Lemma 3.4. (a) *In the above notations we have σ occurs in $\mathscr{F}(n;p,q)$ if and only if $r,s\leqslant n$, and σ' occurs in $\mathscr{F}(n;p,q)$ if and only if $r'\leqslant 2p, s'\leqslant 2q$. If the conditions are* *satisfied then*

$$degree(\sigma) = a_1 + \cdots + a_r + b_1 + \cdots + b_s,$$

$$degree(\sigma') = a'_1 + \cdots + a'_{r'} + b'_1 + \cdots + b'_{s'}. \tag{3.5}$$

(b) *σ occurs in the space of joint harmonics if and only if $r + s \leqslant n$, and σ' occurs in the space of joint harmonics if and only if $r' \leqslant p, s' \leqslant q$. Then we have $\sigma \leftrightarrow \sigma'$ if and only if $r = r',\ s = s',\ a_i = a'_i,\ b_j = b'_j$ for all i, j.*

Definition 3.6. Let $\rho = \rho(\lambda, \Psi)$ be a limit of discrete series representation of $O^*(2n)$. Write λ as in (2.4). Let

$$p = p(\lambda) = m_1 + \cdots + m_k \quad \text{and} \quad q = q(\lambda) = n_1 + \ldots + n_k. \tag{3.7}$$

Notice that $p + q = n$ or $n - 1$.

We define $\Gamma\rho = \rho(\Gamma\lambda, \Gamma\Psi)$ to be the limit or discrete series representation of $Sp(p, q)$ given by

$$\Gamma\lambda = (\overbrace{a_1, \ldots, a_1}^{m_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k}; \overbrace{a_1, \ldots, a_1}^{n_1}, \ldots, \overbrace{a_k, \ldots, a_k}^{n_k}) \tag{3.8}$$

and $\Gamma\Psi$ the unique system of positive roots containing Δ_c^+, with respect to which $\Gamma\lambda$ is dominant, satisfying condition F-1, and such that for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$,

$$e_i - f_j \in \Gamma\Psi \ \Leftrightarrow \ e_i + e_{n-j+1} \in \Psi. \tag{3.9}$$

Lemma 3.10. *Let $\rho = \rho(\lambda, \Psi)$ be a limit of discrete series representation of $O^*(2n)$, $p = p(\lambda)$, and $q = q(\lambda)$. Then the lowest K-types of ρ and $\Gamma\rho$ occur and correspond in the space of joint harmonics for the dual pair $(Sp(p, q), O^*(2n))$.*

Proof. For $1 \leqslant i \leqslant p$, let $\beta_i = \#\{j \leqslant q | e_i - f_j \in \Gamma\Psi\}$, and for $1 \leqslant i \leqslant q$, let $\gamma_i = \#\{j \leqslant p | f_i - e_j \in \Gamma\Psi\}$. Rewrite $\Gamma\lambda = (b_1, \ldots, b_p; c_1, \ldots, c_q)$ and $\lambda = (b_1, \ldots, b_p, -c_q, \ldots, -c_1)$, or with a zero between b_p and $-c_q$. Let σ and σ' be the LKT's of ρ and $\Gamma\rho$, respectively. Then using the formula given in Section 2.3, we get

$$\sigma' = (b'_1, \ldots, b'_p; c'_1, \ldots, c'_q) \ \textit{with} \tag{3.11}$$

$$b'_i = b_i + \beta_i - p + i - 1, \tag{3.12}$$

$$c'_i = c_i + \gamma_i - q + i - 1, \quad \textit{and} \tag{3.13}$$

$$\sigma = (b'_1, \ldots, b'_p, -c'_q, \ldots, -c'_1) + (p - q, \ldots, p - q), \tag{3.14}$$

or with a zero between b'_p and $-c'_q$. Since $b_p \geqslant 1$ and $c_q \geqslant 1$, we have that $b'_p, c'_q \geqslant 0$, and the result follows from Lemma 3.4. $\square$

Theorem 3.15. *Let $\pi = \pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $O^*(2n)$, and let $X(\gamma_{\mathrm{d}})$ be the corresponding standard module induced from discrete series. Let $p = p(\lambda) + r$ and $q = q(\lambda) + r$, so that $p + q = n$ or $n - 1$.*

(a) *The lowest K-types of $X(\gamma_{\mathrm{d}})$ are of minimal degree and occur in the space $\mathscr{H}$ of joint harmonics for the dual pair $(Sp(p, q), O^*(2n))$.*

(b) *Let $\pi' = \pi(r, \Gamma\lambda, \Gamma\Psi, \mu, \nu)$, a representation of $Sp(p, q)$, and let $X(\gamma'_{\mathrm{d}})$ be the corresponding standard module induced from discrete series. Then the lowest K-types of $X(\gamma'_{\mathrm{d}})$ occur in $\mathscr{H}$ and correspond to the lowest K-types of $X(\gamma_{\mathrm{d}})$.*

Proof. If

$$\bar{\gamma}_{\mathrm{d}}|_{\mathrm{t}} = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \tag{3.16}$$

or with one zero in the middle (this is the parameter λ' of (2.12)) then

$$\bar{\gamma}'_{\mathrm{d}}|_{\mathrm{t}} = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}; \overbrace{a_1, \dots, a_1}^{n_1}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}). \tag{3.17}$$

Using formulas (2.15)–(2.18), the LKTs of $X(\gamma_{\mathrm{d}})$ are those of the form

$$\begin{aligned} &(p - q, \dots, p - q) \\ &+ (\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}, \overbrace{-c_k, \dots, -c_k}^{n_k}, \dots, \overbrace{-c_1, \dots, -c_1}^{n_1}) \end{aligned} \tag{3.18}$$

or with one zero in the middle, and where

$$\begin{aligned} b_j &= a_j + R_j - S_j - \tfrac{1}{2}(m_j - n_j + 1) - p + q + \varepsilon_j, \\ c_j &= a_j - R_j + S_j - \tfrac{1}{2}(n_j - m_j + 1) + p - q - \varepsilon_j. \end{aligned} \tag{3.19}$$

The R_j and S_j are as in (2.14), so that $R_k = p$ and $S_k = q$, and for each j, $\varepsilon_j \in \{0, \pm\frac{1}{2}\}$, and is chosen so that b_j and c_j are integers. All such choices for the ε_j yield all the LKTs for $X(\gamma_{\mathrm{d}})$.

Similarly, the LKTs of $X(\gamma'_{\mathrm{d}})$ are those of the form

$$(\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}; \overbrace{c_1, \dots, c_1}^{n_1}, \dots, \overbrace{c_k, \dots, c_k}^{n_k}) \tag{3.20}$$

with the b_j and c_j as in (3.19), and the ε_j subject to the same conditions as for $O^*(2n)$. Since $b_k, c_k \geqslant 0$, these K-types occur and correspond in $\mathscr{H}$ by Lemma 3.4, proving (b) and the second part of (a).

To estimate the degree of K-types in $X(\gamma_\mathrm{d})$, first notice that the LKTs all have the same degree, namely

$$\sum_{i=1}^{k} m_i b_i + \sum_{i=1}^{k} n_i c_i, \tag{3.21}$$

which is independent of the ε_i chosen. Let τ be a K-type occurring in $X(\gamma_\mathrm{d})$. By the standard theory of [12,22] (see e.g., [17, Lemma 5.1.1] for details), τ is of the form

$$\tau = \sigma + \sum_{\alpha} n_\alpha \alpha, \tag{3.22}$$

where σ is a LKT of $X(\gamma_\mathrm{d})$, the sum runs over roots in $\Delta(\mathfrak{l}:\mathfrak{t}) \cup \Delta(\mathfrak{u} \cap \mathfrak{p})$, and $n_\alpha \geqslant 0$ for all α. Here $\mathfrak{l}$ and $\mathfrak{u}$ are as in Section 2.3. If $\bar{\gamma}_\mathrm{d}|_\mathfrak{t}$ is as in (3.16) then the roots in $\Delta(\mathfrak{l}:\mathfrak{t}) \cup \Delta(\mathfrak{u} \cap \mathfrak{p})$ are of the form

$$\pm(e_i - e_j), \quad 1 \leqslant i < j \leqslant p, \tag{3.23}$$

$$\pm(e_{n-j+1} - e_{n-i+1}), \quad 1 \leqslant i < j \leqslant q, \tag{3.24}$$

$$\pm(e_i + e_{n-j+1}), \quad 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q, \tag{3.25}$$

$$(e_i + e_j), \quad 1 \leqslant i < j \leqslant p, \tag{3.26}$$

$$-(e_{n-j+1} + e_{n-i+1}), \quad 1 \leqslant i < j \leqslant q. \tag{3.27}$$

Rewrite

$$\sigma = (p-q, \dots, p-q) + (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \tag{3.28}$$

or with one zero between α_p and β_1. Recall that $\alpha_p \geqslant 0 \geqslant \beta_1$. Then

$$\tau = \sigma + (x_1, \dots, x_p, y_1, \dots, y_q) \tag{3.29}$$

or with one more coordinate z between x_p and y_1. Then

$$\begin{aligned} degree(\tau) &= \sum_{i=1}^{p} |\alpha_i + x_i| + \sum_{i=1}^{q} |\beta_i + y_i| \quad (+|z|) \\ &\geqslant \sum_{i=1}^{p} (\alpha_i + x_i) - \sum_{i=1}^{q} (\beta_i + y_i) \end{aligned}$$

$$= \sum_{i=1}^{p} \alpha_i - \sum_{i=1}^{q} \beta_i + \sum_{i=1}^{p} x_i - \sum_{i=1}^{q} y_i$$

$$= degree(\sigma) + \sum_{i=1}^{p} x_i - \sum_{i=1}^{q} y_i. \tag{3.30}$$

But $\sum_{i=1}^{p} x_i - \sum_{i=1}^{q} y_i \geqslant 0$ since each of the roots contributing to the sum (3.22) is of the form (3.23)–(3.27). This finishes the proof of the theorem. □

3.2. General facts about the correspondence

We now state some (mostly standard) results about how the correspondences for different dual pairs of the form $(Sp(p,q), O^*(2n))$ are related.

Recall first how the groups $Sp(p,q)$ and $O^*(2n)$ are embedded in $Sp(4n(p+q),\mathbb{R})$ (see [7,20]). Let V be a $(p+q)$-dimensional (right) vector space over $\mathbb{H}$, with hermitian form $(,)$ of signature (p,q), $\mathbb{H}$-linear in the second variable, and let V' be an n-dimensional (left) vector space over $\mathbb{H}$, with skew-hermitian form $(,)'$ which is $\mathbb{H}$-linear in the first variable. Now define a symplectic space $(W, \langle , \rangle)$ as follows: $W = V \otimes_{\mathbb{H}} V'$ as a vector space over $\mathbb{R}$, and $\langle , \rangle = tr_{\mathbb{H}/\mathbb{R}}((,) \otimes \overline{(,)'})$. Then the isometry groups of $(,)$, $(,)'$, and $\langle , \rangle$ are isomorphic to $Sp(p,q)$, $O^*(2n)$, and $Sp(4n(p+q),\mathbb{R})$ respectively, and this defines an embedding of the first two groups as a reductive dual pair in the symplectic group.

Notation 3.31. Let π and π' be irreducible admissible representations of $Sp(p,q)$ and $O^*(2n)$ respectively. If $\pi \leftrightarrow \pi'$ in the correspondence for the dual pair $(Sp(p,q), O^*(2n))$ we write $\theta_n(\pi) = \pi'$ and $\theta_{p,q}(\pi') = \pi$. If π does not occur in the correspondence, we write $\theta_n(\pi) = 0$, and similarly for π'. If there is no confusion possible about the dual pair under consideration, we will omit the subscript and write $\theta(\pi) = \pi'$, etc.

If π is an irreducible admissible representation of G, let π^* denote the contragredient representation. (Notice that if $G = Sp(p,q)$, then $\pi^* = \pi$.) The following result is due to Przebinda ([18, Theorem 5.5]).

Lemma 3.32. *Let π be an irreducible admissible representation of $O^*(2n)$. Then $\theta_{p,q}(\pi)^* = \theta_{q,p}(\pi^*)$.*

Let V_1 and V_2 be $\mathbb{H}$-hermitian or $\mathbb{H}$-skew-hermitian spaces. Then $V = V_1 \oplus V_2$ will be a space of the same type. This defines natural embeddings $Sp(p,q) \times Sp(r,s) \to Sp(p+r, q+s)$ and $O^*(2n) \times O^*(2m) \to O^*(2(n+m))$, so that any representation of $Sp(p+r, q+s)$ is a representation of $Sp(p,q) \times Sp(r,s)$ by restriction. Similarly for $O^*(2(n+m))$.

If p, q, and n are non-negative integers, we denote the oscillator representation for the dual pair $(Sp(p,q), O^*(2n))\omega_{p,q,n}$ (this is a representation of $\widetilde{Sp}(2n(p+q),\mathbb{R})$).

Lemma 3.33. *Let p, q, r, s, n, and m be non-negative integers. Then*

1. $\omega_{p,q,n} \otimes \omega_{r,s,n} \cong \omega_{p+r,q+s,n}$ *as representations of* $Sp(p,q) \times Sp(r,s) \times O^*(2n)$, *with* $O^*(2n)$ *acting diagonally on the left-hand side; and*
2. $\omega_{p,q,n} \otimes \omega_{p,q,m} \cong \omega_{p,q,n+m}$ *as representations of* $Sp(p,q) \times O^*(2n) \times O^*(2m)$, *with* $Sp(p,q)$ *acting diagonally on the left-hand side.*

Proof. If W_1 and W_2 are symplectic spaces with isometry groups Sp_1 and Sp_2 respectively, then $W = W_1 \oplus W_2$ is a symplectic space, and this defines an embedding

$$Sp_1 \times Sp_2 \to Sp = Sp(W). \tag{3.34}$$

By Corollary 5.6 of [19], this embedding lifts to a map $\beta : \widetilde{Sp}_1 \times \widetilde{Sp}_2 \to \widetilde{Sp}$ of the metaplectic covers so that for the corresponding oscillator representations, we have

$$\omega_1 \otimes \omega_2 \cong \omega \circ \beta \tag{3.35}$$

as representations of $\widetilde{Sp}_1 \times \widetilde{Sp}_2$.

Let V_1, V_2 be hermitian spaces over $\mathbb{H}$, and let V' be a skew-hermitian space. Let $G(V_i)$ and $G(V')$ be the corresponding isometry groups. Write

$$W = (V_1 \oplus V_2) \otimes V' \cong (V_1 \otimes V') \oplus (V_2 \otimes V') = W_1 \oplus W_2 \tag{3.36}$$

(as vector spaces over $\mathbb{R}$). Then we get corresponding embeddings

$$G(V_1) \times G(V_2) \times G(V') \to \widetilde{Sp} \tag{3.37}$$

and

$$G(V_1) \times G(V') \times G(V_2) \times G(V') \to \widetilde{Sp}_1 \times \widetilde{Sp}_2.$$

The action of $G(V')$ on $(V_1 \oplus V_2) \otimes V'$ corresponds under isomorphism (3.36) to the diagonal action on $(V_1 \otimes V') \oplus (V_2 \otimes V')$. Restricting $\omega_1 \otimes \omega_2$ and ω in (3.35) to these subgroups now yields the first part of the lemma. The second part is similar. □

The next result rules out that a representation of $O^*(2n)$ lifts to more than one group of the form $Sp(p,q)$ with $p+q \leqslant n$, so that our result in Theorem 5.1 indeed gives us all such occurrences.

Proposition 3.38. *Let* $\pi \in \widehat{O^*(2n)}$. *Suppose* $\theta_{p,q}(\pi) \neq 0$ *and* $\theta_{r,s}(\pi) \neq 0$. *Then either*

(a) $p - r = q - s$, *i.e.*, $Sp(p,q)$ *and* $Sp(r,s)$ *are in the same Witt Tower; or*
(b) $p - r \neq q - s$, $p + s \geqslant n$ *and* $q + r \geqslant n$, *and in particular*, $p + q + r + s > 2n$.

Proof. Assume $\theta_{p,q}(\pi)\neq 0$ and $\theta_{r,s}(\pi)\neq 0$. By Lemma 3.32, $\theta_{s,r}(\pi^*)\neq 0$ as well. So π and π^* are quotients of $\omega_{p,q,n}$ and $\omega_{s,r,n}$, respectively. By Lemma 3.33(1), $\pi\otimes\pi^*$, and therefore the trivial representation $\mathbb{1}$ of $O^*(2n)$ is a quotient of $\omega_{p+s,q+r,n}$, i.e., $\theta_{p+s,q+r}(\mathbb{1})\neq 0$. So the trivial K-type must occur in the space of joint harmonics for the dual pair $(Sp(p+s,q+r),O^*(2n))$. The conditions on p,q,r,s given in the proposition now follow from Lemma 3.4. $\square$

4. The induction principle

We formulate the induction principle, which is due to Kudla [13] (see also [1,2,16,17]), for the dual pairs $(Sp(p,q),O^*(2n))$.

Let $D=\mathbb{H}$ with the standard involution.

Let V (resp. V') be a right (resp., left) vector space over D equipped with a hermitian form $(,)$ (resp. a skew-hermitian form $(,)'$). Denote by $G(V)$ (resp. $G(V')$) the isometry group. Then $W=V\otimes_D V'$ is a real symplectic space with the symplectic form $tr((,)\otimes\overline{(,)'})$, where tr is the reduced trace map from D to $\mathbb{R}$. Furthermore $(G(V),G(V'))$ is a reductive dual pair in $Sp(W)$.

Suppose that

$$V=V_+\oplus V_0\oplus V_-, \tag{4.1}$$

$$V'=V'_+\oplus V'_0\oplus V'_-, \tag{4.2}$$

where V_+, V_- (resp. V'_+, V'_-) are totally isotropic subspaces and are dual to each other with respect to $(,)$ (resp. $(,)'$).

Let $W_0=V_0\otimes V'_0$, and let $W_0=X_0\oplus Y_0$ be a complete polarization. Then W admits the following complete polarization:

$$W=X\oplus Y, \tag{4.3}$$

where

$$X=(V\otimes V'_+)\oplus(V_+\otimes V'_0)\oplus X_0, \tag{4.4}$$

$$Y=(V\otimes V'_-)\oplus(V_-\otimes V'_0)\oplus Y_0. \tag{4.5}$$

Let $P_V=P(V_+)$ (resp. $P_{V'}=P(V'_+)$) be the stabilizer of V_+ in $G(V)$ (resp. V'_+ in $G(V')$). We have the Levi decomposition

$$P_V=M_VN_V,\quad P_{V'}=M_{V'}N_{V'}, \tag{4.6}$$

where

$$M_V\cong GL(V_+)\times G(V_0),\quad M_{V'}\cong GL(V'_+)\times G(V'_0). \tag{4.7}$$

Define the real symplectic space

$$W_M = (V_- \otimes V'_+) \oplus (V_+ \otimes V'_-) \oplus W_0. \tag{4.8}$$

Then $(M_V, M_{V'})$ is a dual pair in $Sp(W_M)$. Note that W_M has the following complete polarization:

$$W_M = X_M \oplus Y_M, \tag{4.9}$$

where

$$X_M = (V_- \otimes V'_+) \oplus X_0, \tag{4.10}$$

$$Y_M = (V_+ \otimes V'_-) \oplus Y_0. \tag{4.11}$$

The oscillator representation ω associated to the dual pair $(G(V), G(V')) \subseteq Sp(W)$ may be realized on the Schwartz space $\mathscr{S}(Y)$, and the oscillator representation ω_M associated to the dual pair $(M_V, M_{V'}) \subseteq Sp(W_M)$ may be realized on the Schwartz space $\mathscr{S}(Y_M)$ (see [19]).

Let ρ: $\mathscr{S}(Y) \to \mathscr{S}(Y_M)$ be the obvious restriction map.

Denote $d = dim_{\mathbb{R}}\, D = 4$, and $d_0 = dim_{\mathbb{R}}\{t \in D | \bar{t} = -t\} = 3$.

We set

$$m_0 = dim_D\, V_0, \quad n_0 = dim_D\, V'_0, \quad k = dim_D\, V_+, \quad l = dim_D\, V'_+. \tag{4.12}$$

For $A \in M_{k\times k}(D)$, let $det(A)$ be the usual determinant of A realized as an element of $\mathfrak{u}^*(2k) \subset M_{2k\times 2k}(\mathbb{C})$. (If k = 1 then this is the reduced norm as in Section 2.2.)

Let ξ be the following character of $GL(V_+) \times GL(W_+)$:

$$\xi(h, h') = det(h)^{(n_0+l)d}\, det(h')^{(m_0+k)d}, \quad (h, h') \in GL(V_+) \times GL(V'_+). \tag{4.13}$$

By comparing the actions of the groups involved in the mixed models of ω and ω_M (c.f. [17] for the case $D = \mathbb{C}$), we have

Proposition 4.14. *The restriction map ρ is a surjective $P_V \times P_{V'}$ equivariant map*

$$\omega \to \omega_M \otimes \xi. \tag{4.15}$$

Let $\mathfrak{n}_V$ (resp. $\mathfrak{n}_{V'}$) be the Lie algebra of N_V (resp. $N_{V'}$), and let $\rho(\mathfrak{n}_V)$ (resp. $\rho(\mathfrak{n}_{V'})$) be half the sum of the roots of $\mathfrak{n}_V$ (resp. $\mathfrak{n}_{V'}$) with respect to a Cartan subgroup of M_V (resp. $M_{V'}$). Then $\rho(\mathfrak{n}_V)$ (resp. $\rho(\mathfrak{n}_{V'})$) exponentiates to a character ρ_V of P_V (resp. $\rho_{V'}$ of $P_{V'}$). Notice that $N = N_V$ and $N' = N_{V'}$ admit exact sequences

$$1 \to ZN \to N \to Hom(V_0, V_+) \to 1, \tag{4.16}$$

$$1 \to ZN' \to N' \to Hom(V'_0, V'_+) \to 1, \tag{4.17}$$

where ZN and ZN' are the centers of N and N', respectively. Furthermore, $ZN \cong B(V_-)_{\text{skew}}$, the space of skew-hermitian forms on V_-, and $ZN' \cong B(V'_-)_{\text{herm}}$, the space of hermitian forms on V'_-. We therefore see that the characters ρ_V and $\rho_{V'}$ are given by

$$\rho_V(h, g_0, n_V) = det(h)^{\frac{1}{4}(m_0 d + (k-1)d + 2d_0)} \tag{4.18}$$

for $h \in GL(V_+),\ g_0 \in G(V_0),\ n_V \in N_V$ and

$$\rho_{V'}(h', g'_0, n_{V'}) = det(h')^{\frac{1}{4}(n_0 d + (l+1)d - 2d_0)} \tag{4.19}$$

for $h' \in GL(V'_+),\ g'_0 \in G(V'_0),\ n_{V'} \in N_{V'}$. They are the modulus functions of P_V and $P_{V'}$.

We now state the Induction Principle which follows from Proposition 4.14 using Frobenius Reciprocity.

Theorem 4.20. *Let $\pi \in \widehat{G(V_0)},\ \pi' \in \widehat{G(V'_0)},\ \sigma \in \widehat{GL(V_+)}$, and $\sigma' \in \widehat{GL(V'_+)}$. Suppose that $\pi \leftrightarrow \pi'$, and $\sigma \leftrightarrow \sigma'$ in the correspondence of dual pairs $(G(V_0), G(V'_0))$ and $(GL(V_+), GL(V'_+))$. Let χ_V and $\chi_{V'}$ be the characters of $GL(V_+)$ and $GL(V'_+)$ given by*

$$\chi_V(h) = det(h)^{\frac{1}{4}((n_0 + l - m_0 - k + 1)d - 2d_0)}, \quad h \in GL(V_+), \tag{4.21}$$

$$\chi_{V'}(h') = det(h')^{\frac{1}{4}((m_0 + k - n_0 - l - 1)d + 2d_0)}, \quad h' \in GL(V'_+). \tag{4.22}$$

Then there is a non-zero $G(V) \times G(V')$-map

$$\Psi : \omega \to Ind_{P_V}^{G(V)}(\pi \otimes \sigma \otimes \chi_V) \otimes Ind_{P_{V'}}^{G(V')}(\pi' \otimes \sigma' \otimes \chi_{V'}). \tag{4.23}$$

The following theorem is the extended induction principle which is due to Adams and Barbasch [1].

Theorem 4.24. *Let Ψ be as in Theorem 4.20, $G = G(V)$, $G' = G(V')$, $M = M_V$, and $M' = M_{V'}$, and let K and K' be maximal compact subgroups of G and G', respectively. Suppose μ is a K-type and λ is a $(K \cap M)$-type such that the following conditions are satisfied:*

1. *λ occurs and is of minimal degree in $\pi \otimes \sigma$.*
2. *μ occurs and is of minimal degree and of multiplicity one in $Ind_{P_V}^{G}(\pi \otimes \sigma \otimes \chi_V)$.*
3. *μ and λ have the same degree, and the restriction of μ to $K \cap M$ contains λ.*
4. *There exist characters α and α' of M and M' which are trivial on $K \cap M$ and $K' \cap M'$, such that $(\pi \otimes \sigma \otimes \alpha) \otimes (\pi' \otimes \sigma' \otimes \alpha')$ is a quotient of ω_M, and $Ind_{P_V}^{G}(\pi \otimes \sigma \otimes \alpha \otimes \chi_V)$ is irreducible.*

Let μ' be the K'-type which corresponds to μ in the space of joint harmonics. Then $\mu \otimes \mu'$ is in the image of Ψ.

The statement of the theorem is also true with the roles of V and V' reversed.

We are going to apply these theorems to the following situation: $G(V) = Sp(p,q)$, $G(V') = O^*(2n)$, $M = Sp(p-1,q-1) \times GL(1,\mathbb{H})$, and $M' = O^*(2(n-2)) \times GL(1,\mathbb{H})$. The characters χ_V and $\chi_{V'}$ then become $\chi_V(h) = det(h)^{n-p-q-\frac{1}{2}}$, and $\chi_{V'}(h) = det(h)^{p+q-n+\frac{1}{2}}$, so that $\chi_{V'} = \chi_V^*$. Notice that in our parametrization of Section 2.2, $\chi_V = \sigma(1, 2(n-p-q)-1)$ and $\chi_{V'} = \sigma(1, 2(p+q-n)+1)$.

We will need to know something about the correspondence for the dual pairs $(GL(n,\mathbb{H}), GL(m,\mathbb{H}))$. We first describe the space of joint harmonics.

Proposition 4.25. *Let $n \leqslant m$. The correspondence of $Sp(n)$- and $Sp(m)$-types in the space $\mathscr{H}$ of joint harmonics for the dual pair $(GL(n,\mathbb{H}), GL(m,\mathbb{H}))$ is given as follows. If $\sigma = (a_1, \dots, a_n)$ is an $Sp(n)$-type, then σ occurs in $\mathscr{H}$, and $\sigma \leftrightarrow \sigma'$, where $\sigma' = (a_1, \dots, a_n, 0, \dots, 0)$. The degree of σ and σ' is $\sum_{i=1}^n a_i$.*

Proof. Consider the diamond dual pairs (see [10])

$$\begin{array}{ccccccccccc} & & O^*(4n) & & & & & & Sp(m) & & \\ & / & & \backslash & & & & / & & \backslash & \\ GL(n,\mathbb{H}) & & & & U(2n) & & GL(m,\mathbb{H}) & & & & U(2m) \\ & \backslash & & / & & & & \backslash & & / & \\ & & Sp(n) & & & & & & O^*(4m) & & \end{array} \tag{4.26}$$

Recall that any two groups positioned at corresponding corners of the two diamonds are a dual pair in $Sp(8nm,\mathbb{R})$. Using the known correspondence of K-types for the three dual pairs $(O^*(4n), Sp(m))$, $(U(2n), U(2m))$, and $(Sp(n), O^*(4m))$ and the theory of [10, Section 3], the correspondence for $(GL(n,\mathbb{H}), GL(m,\mathbb{H}))$ may now easily be obtained. $\square$

Proposition 4.27. *Let $\sigma = \sigma(\mu, v)$ be an irreducible representation of $GL(1,\mathbb{H})$ (see Section 2.2). Then σ occurs in the correspondence for the dual pair $(GL(1,\mathbb{H}), GL(1,\mathbb{H}))$, and $\theta(\sigma) = \sigma^*$, i.e., $\theta(\sigma)$ is $\sigma(\mu, -v)$. The corresponding $SU(2)$-type has degree $\mu - 1$.*

Proof. Occurrence follows from the results of [1] by considering the see-saw dual pairs

$$\begin{array}{ccc} GL(2,\mathbb{C}) & & GL(1,\mathbb{H}) \\ & \times & \\ GL(1,\mathbb{H}) & & GL(1,\mathbb{C}). \end{array} \tag{4.28}$$

Let $k = (\mu, 0)$, $z = (\nu, 0)$, and let $L(k, z)$ be the representation of $GL(2, \mathbb{C})$ determined by k and z as in Section 1 of [1]. Recall that $L(k, z)$ is the unique irreducible quotient of an induced representation $Ind_P^{GL(2,\mathbb{C})}(\chi_{k,z})$, where $P = MN$ is a parabolic subgroup of $GL(2, \mathbb{C})$ with Levi factor $GL(1, \mathbb{C}) \times GL(1, \mathbb{C})$ and $\chi_{k,z}$ is the character of $GL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \cong (\mathbb{C}^\times)^2$ given by $\chi_{k,z}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = r_1^\nu e^{i\mu\theta_1}$. Then $L(k, z)$ occurs in the correspondence with $GL(1, \mathbb{C})$ (it lifts to $L(-\mu, -\nu) \in \widehat{GL(1, \mathbb{C})}$). Since the restriction of $L(k, z)$ to $GL(1, \mathbb{H})$ contains σ as a constituent, σ occurs in the correspondence for the dual pair $(GL(1, \mathbb{H}), GL(1, \mathbb{H}))$. The theta lift can be determined by looking at the way the members of the dual pair are embedded in $Sp(8, \mathbb{R})$ (see [6]): Write $W = X \oplus Y$ a complete polarization of the eight-dimensional symplectic space W. Think of X and Y as two-dimensional complex vector spaces, and identify $GL(1, \mathbb{H})$ with $U^*(2) \subset GL(2, \mathbb{C})$. If $(g, h) \in GL(1, \mathbb{H}) \times GL(1, \mathbb{H})$ and $(x, y) \in W$, then $(g, h) \cdot (x, y) = (gxh^{-1}, hxg^{-1})$. Consequently, the two factors will act by contragredients. The degree of the $SU(2)$-type with highest weight $(\frac{\mu-1}{2}, -\frac{\mu-1}{2})$ (this is the $Sp(1)$-type with highest weight $(\mu - 1)$) is given by Proposition 4.25. $\square$

5. The main theorems

5.1. The cases $p + q = n$ or $n - 1$

We are now ready to state and prove our main theorem.

Theorem 5.1. *Let $\pi' = \pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $G' = O^*(2n)$. Let $p = p(\lambda) + r$, and $q = q(\lambda) + r$ (so that $p + q = n$ or $n - 1$), and let $\pi = \pi(r, \Gamma\lambda, \Gamma\Psi, \mu, \nu)$, a representation of $G = Sp(p, q)$ (see Definition 3.6). Then*

1. $\theta_{p,q}(\pi') = \pi$.
2. *The lowest K'-types of π' are of minimal degree in π' and correspond in $\mathscr{H}$ to the lowest K-types of π.*
3. *If p' and q' are integers so that $p' + q' = n$ or $n - 1$ and $(p', q') \neq (p, q)$ then $\theta_{p',q'}(\pi') = 0$; i.e., part 1 completely describes the duality correspondence for the dual pairs $(Sp(p, q), O^*(2n))$ with $p + q = n$ or $n - 1$.*

Remark 5.2. The algorithm of Definition 3.6 is easy to reverse, and it follows that if $p + q = n$ then every irreducible admissible representation of $Sp(p, q)$ has non-zero theta lifts to both $O^*(2n)$ and $O^*(2n + 2)$ Theorem follows.

Proof. To prove Part 1, we use induction on n, the rank of $O^*(2n)$. The base case is when π' is a discrete series. Part 1 in this case follows from Theorem 6.2 of [15], and Part 2 from Lemma 3.10 and Theorem 3.15.

So now assume that π' is not a discrete series representation. Then there are a representation π'_0 of $O^*(2(n-2))$ and a representation σ of $GL(1, \mathbb{H})$ such that π' is a lowest K'-type constituent of $I' = Ind_{P'}^{G'}(\pi'_0 \otimes \sigma)$, where $P' = M'N'$ is a parabolic subgroup of G' with Levi factor $M' \cong O^*(2(n-2)) \times GL(1, \mathbb{H})$. We can choose σ (if necessary replace by its contragredient) so that π' is a quotient of the induced representation. Let $\pi_0 = \theta_{p-1,q-1}(\pi'_0)$ (by the induction hypothesis). (For the cases $n=2$ and $n=3$ we formally define the correspondences for the dual pairs $(Sp(0,0), O^*(0))$ and $(Sp(0,0), O^*(2))$ to be $\mathbb{1} \leftrightarrow \mathbb{1}$; here $\mathbb{1}$ is the trivial representation of the appropriate group.) Then it is easy to compare Langlands parameters and see that π is a lowest K-type constituent of $I = Ind_P^G(\pi_0 \otimes \sigma^*)$, where $P = MN$ is a parabolic subgroup of G with Levi factor $M \cong Sp(p-1, q-1) \times GL(1, \mathbb{H})$. (Recall from Section 2.2 that if v' is obtained from v by changing some of the signs of the coefficients, then $\pi(r, \lambda, \Psi, \mu, v') \cong \pi(r, \lambda, \Psi, \mu, v)$). By Theorem 4.20 and Proposition 4.27, there is a non-zero $G \times G'$ map Φ from the oscillator representation ω for the dual pair (G, G') to $I \otimes I'$. By Theorem 3.15, we know that every lowest K'-type η' of π' is of minimal degree in I' and corresponds in $\mathscr{H}$ to a lowest K-type η of the standard module $X(\gamma_{\mathrm{d}})$ induced from discrete series which contains π as a lowest K-type constituent. Moreover, η' has multiplicity one in I'. To prove Part 1, we will use Theorem 4.24, with the roles of G and G' interchanged, to show that $\eta \otimes \eta'$ is in the image of Φ, and then show that η is a lowest K-type of π. Since η then has multiplicity one in I, $\pi \otimes \pi'$ must then be a quotient of $I \otimes I'$, and hence of ω. This will also finish the proof of Part 2.

Write $\gamma' = (\Psi, \Gamma', \bar{\gamma}')$ and $\gamma'_0 = (\Psi_0, \Gamma'_0, \bar{\gamma}'_0)$ for the final limit characters for π' and π'_0 respectively, and let $\mathfrak{t}'$ and $\mathfrak{t}^0$ be compact Cartan subalgebras of $O^*(2n)$ and $O^*(2(n-2))$, respectively, with $\mathfrak{t}^0 \subset \mathfrak{t}'$. If

$$\bar{\gamma}'|_{\mathfrak{t}'} = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \tag{5.3}$$

or with one zero in the middle, then σ will be of the form $(2a_j, z)$ for some j and some complex number z, and

$$\begin{aligned} \bar{\gamma}'_0|_{\mathfrak{t}^0} = (&\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_j, \dots, a_j}^{m_j-1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \\ &\overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_j, \dots, -a_j}^{n_j-1}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \end{aligned} \tag{5.4}$$

or with one zero in the middle. Let η' be a lowest K'-type of π', and let ζ be the $(K' \cap M')$-type with highest weight equal to the restriction of (the highest weight of) η' to $\mathfrak{t}^0$. Write (as in the proof of Theorem 3.15)

$$\begin{aligned} \eta' = &(p-q, \dots, p-q) \\ &+ (\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}, \overbrace{-c_k, \dots, -c_k}^{n_k}, \dots, \overbrace{-c_1, \dots, -c_1}^{n_1}), \end{aligned} \tag{5.5}$$

or with one zero in the middle, and with the b_i and c_i as in (3.19). Then it is easy to see that $\zeta = \beta + \xi$, where

$$\beta = (p-q,\dots,p-q) + (\overbrace{b_1,\dots,b_1}^{m_1},\dots,\overbrace{b_j,\dots,b_j}^{m_j-1},\dots,\overbrace{b_k,\dots,b_k}^{m_k},$$

$$\overbrace{-c_k,\dots,-c_k}^{n_k},\dots,\overbrace{-c_j,\dots,-c_j}^{n_j-1},\dots,\overbrace{-c_1,\dots,-c_1}^{n_1}) \tag{5.6}$$

or with one zero in the middle, and $\xi = (\frac{2a_j-1}{2}, -\frac{2a_j-1}{2})$ is the highest weight of the $SU(2)$-type of σ. Notice that β is a lowest $U(n-2)$-type of π'_0, hence by Theorem 3.15 of minimal degree in π'_0. Also, ξ is clearly of minimal degree in σ since σ contains only one $SU(2)$-type. So η' and ζ will play the roles of μ and λ in Theorem 4.24. We can read the degrees of η' and β off formulas (5.5) and (5.6), and using (3.19) and Proposition 4.25 we have

$$degree(\eta') - degree(\beta) = b_j + c_j = 2a_j - 1 = degree(\xi). \tag{5.7}$$

Consequently, η' and ζ satisfy parts (1) through (3) of Theorem 4.24. For part (4), notice that if we choose α to be a character of $GL(1,\mathbb{H})$ of the form $(det)^w$ for some complex number w, and $\alpha' = \alpha^*$, then α and α' will be trivial on $K\cap M$ and $K'\cap M'$, and if ρ and ρ' are representations of M and M' so that $\rho\otimes\rho'$ is a quotient of ω_M, then $(\rho\otimes\alpha)\otimes(\rho'\otimes\alpha')$ will be a quotient of ω_M as well. It remains to show that w can be chosen so that $Ind_P^G(\pi\otimes\sigma\otimes\alpha)$ is irreducible. But this follows from [21].

Now let η be the K-type which corresponds to η' in $\mathscr{H}$. To finish the proof of the theorem, we must show that η is a lowest K-type of π. If π' is a limit of discrete series representation, then this follows from Lemma 3.10. If not, then we may assume that our representation $\sigma = \sigma(2a_j, z)$ satisfies the following condition: $z\neq 0$ or $2a_j$ is odd. (This is the non-parity condition of Section 2.2.) In this case, the induced representation I has only one lowest K-type constituent, namely π. Since (as is easily checked) η is a lowest K-type of the standard module $X(\gamma_{\mathrm{d}})$ induced from discrete series which has π as a constituent, and the constituents of I are also constituents of $X(\gamma_{\mathrm{d}})$, η must be a lowest K-type of π. This completes the proof of Parts 1 and 2. Part 3 follows from Proposition 3.38. □

5.2. Going up the Witt towers

Starting from the correspondence for the dual pairs $(Sp(p,q), O^*(2n))$ with $p+q = n, n-1$, we can determine the correspondence for all cases $p+q\leqslant n$, as well as for some of the cases $n<p+q$, by moving up one Witt tower at a time, using the induction principle in a one-sided way.

Theorem 5.8. *Let* $\pi' = \pi(r,\lambda,\Psi,\mu,\nu)\in\widehat{O^*(2n)}$, $p = p(\lambda)+r$, *and* $q = q(\lambda)+r$ (*so that* $p+q = n$ *or* $n-1$). *Let* $\pi = \theta_{p,q}(\pi') = \pi(r,\Gamma\lambda,\Gamma\Psi,\mu,\nu)\in\widehat{Sp(p,q)}$ *be as in*

Theorem 5.1. *Write* $\mu = (\mu_1, \dots, \mu_r)$ *and* $\nu = (\nu_1, \dots, \nu_r)$. *Let* s *be a non-negative integer. Then*

$$\theta_{n+2s}(\pi) = \pi'_s = \pi(r+s, \lambda, \Psi, \mu^s, \nu^s), \tag{5.9}$$

where $\mu^s = (\mu_1, \dots .\mu_r, \underbrace{1, \dots, 1}_{s})$, *and*

$$\nu^s = \begin{cases} (\nu_1, \dots, \nu_r, 1, 5, 9, \dots, 4s-3) & \text{if } p+q = n, \\ (\nu_1, \dots, \nu_r, 3, 7, 11, \dots, 4s-1) & \text{if } p+q = n-1. \end{cases} \tag{5.10}$$

Proof. We use induction on s. The base case $s = 0$ is Theorem 5.1. So assume $s > 0$ and $\theta_{n+2s-2}(\pi) = \pi'_{s-1}$. By Theorem 4.20, there is a nonzero $Sp(p,q) \times O^*(2(n+2s))$ map from the oscillator representation for the dual pair $(Sp(p,q), O^*(2(n+2s)))$ to

$$\pi \otimes Ind_{P'}^{O^*(2(n+2s))}(\pi'_{s-1} \otimes \sigma), \tag{5.11}$$

where $P' = M'N'$ is a parabolic subgroup of $O^*(2(n+2s))$ with Levi factor $M' \cong O^*(2(n+2s-2)) \times GL(1, \mathbb{H})$, and σ is the character of $GL(1, \mathbb{H})$ given by $(1, 4s-3)$ if $p+q = n$, and $(1, 4s-1)$ if $p+q = n-1$. Consequently, $\theta_{n+2s}(\pi)$ is a constituent of this induced representation I'. The representation π'_s is the unique lowest K'-type constituent of I'. Let η be a lowest K-type of π. Then by Theorems 5.1 and 3.1, η is of minimal degree in π. Therefore, if we show that η corresponds in $\mathscr{H}$ to a lowest K'-type η' of π'_s, then we have proved that $\theta_{n+2s}(\pi) = \pi'_s$ (since η' has multiplicity one in I').

Let $\gamma' = (\Psi', \Gamma', \bar{\gamma}')$ and $\gamma'_s = (\Psi'_s, \Gamma'_s, \bar{\gamma}'_s)$ be the final limit characters of π' and π'_s respectively, and let $\mathfrak{t}'$ and $\mathfrak{t}^s$ be compact Cartan subalgebras of the Lie algebras of $O^*(2n)$ and $O^*(2(n+2s))$, respectively, with $\mathfrak{t}' \subset \mathfrak{t}^s$. Write

$$\bar{\gamma}'|_{\mathfrak{t}'} = (\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \tag{5.12}$$

or with one zero in the middle. Then the lowest K-types of π (which by Theorem 5.1 correspond in $\mathscr{H}$ to the lowest K'-types of π') are of the form

$$\eta = (\overbrace{b_1, \dots, b_1}^{m_1}, \dots, \overbrace{b_k, \dots, b_k}^{m_k}; \overbrace{c_1, \dots, c_1}^{n_1}, \dots, \overbrace{c_k, \dots, c_k}^{n_k}), \tag{5.13}$$

with the b_j and c_j as in (3.19). Now

$$\begin{aligned} \bar{\gamma}'_s|_{\mathfrak{t}^s} = (&\overbrace{a_1, \dots, a_1}^{m_1}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}^{s}, \\ &\overbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}^{s}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1}) \end{aligned} \tag{5.14}$$

or with one zero between the $\frac{1}{2}$'s and the $-\frac{1}{2}$'s. Let η' be a lowest K'-type of π'_s. First, assume that $a_k \neq \frac{1}{2}$. Then

$$\eta' = (p-q, \dots, p-q) + (\overbrace{\beta_1, \dots, \beta_1}^{m_1}, \dots, \overbrace{\beta_k, \dots, \beta_k}^{m_k}, \overbrace{z, \dots, z}^{s},$$

$$\overbrace{-w, \dots, -w}^{s}, \overbrace{-\gamma_k, \dots, -\gamma_k}^{n_k}, \dots, \overbrace{-\gamma_1, \dots, -\gamma_1}^{n_1}) \tag{5.15}$$

or with an extra coordinate y between the z's and the $-w$'s, where

$$\begin{aligned} \beta_j &= a_j + R_j - S_j - \tfrac{1}{2}(m_j - n_j + 1) - p + q + \varepsilon'_j, \\ \gamma_j &= a_j - R_j + S_j - \tfrac{1}{2}(n_j - m_j + 1) + p - q - \varepsilon'_j \end{aligned} \tag{5.16}$$

with the R_j and S_j as in (2.14), and $\varepsilon'_j = 0$ or $\pm\frac{1}{2}$ so that β_j and γ_j are integers. Also,

$$\begin{aligned} z &= \tfrac{1}{2} + (R_k + s) - (S_k + s) - \tfrac{1}{2}(s - s + 1) \\ &\quad - p + q + \delta \quad (\delta = 0 \text{ or } \pm\tfrac{1}{2} \text{ as for } \varepsilon'_j \text{ above}) \\ &= \delta \quad \text{since } R_k - S_k = p - q \\ &= 0 \quad \text{since } z \text{ must be an integer.} \end{aligned} \tag{5.17}$$

Similarly, $w = 0$ and $y = 0$.

If $a_k = \frac{1}{2}$, then $m_k = n_k$ since $\frac{1}{2}$ cannot be a coefficient of a discrete series parameter, so

$$\begin{aligned} \eta' &= (p-q, \dots, p-q) \\ &\quad + (\overbrace{\beta_1, \dots, \beta_1}^{m_1}, \dots, \overbrace{\beta_{k-1}, \dots, \beta_{k-1}}^{m_{k-1}}, \overbrace{\beta_k, \dots, \beta_k}^{m_k+s}, \\ &\quad \overbrace{-\gamma_k, \dots, -\gamma_k}^{n_k+s}, \overbrace{-\gamma_{k-1}, \dots, -\gamma_{k-1}}^{n_{k-1}}, \dots, \overbrace{-\gamma_1, \dots, -\gamma_1}^{n_1}) \end{aligned} \tag{5.18}$$

or with one y in the middle, where the β_j and γ_j are as above if $1 \leqslant j \leqslant k-1$, and

$$\begin{aligned} \beta_k &= \tfrac{1}{2} + (R_k + s) - (S_k + s) - \tfrac{1}{2}(m_k + s - (n_k + s)) - p + q + \delta \\ &= 0 \quad \text{since } R_k - S_k = p - q, \ m_k = n_k, \text{ and } \beta_k \in \mathbb{Z}. \end{aligned} \tag{5.19}$$

Similarly, $\gamma_k = 0$ and $y = 0$.

In either case, using Lemma 3.4, we see that η' corresponds in $\mathscr{H}$ to a K-type η_0 which differs from η at most in the choice of the ε_j's, so η_0 is a lowest K-type of π. $\square$

Corollary 5.20. *For $p+q \leqslant n$, every irreducible admissible representation of $Sp(p,q)$ occurs in the theta correspondence for the dual pair $(Sp(p,q), O^*(2n))$.*

Because of Theorem 5.1, we can assign to each irreducible admissible representation π' of $O^*(2n)$ a well-defined signature and rank.

Definition 5.21. (a) We let $sgn(\pi') = (p,q)$ if $\theta_{p,q}(\pi') \neq 0$ and $p+q$ is minimal subject to that condition.

(b) If $sgn(\pi') = (p,q)$ we define $rank(\pi) = p+q$.

Notice that $rank(\pi') \leqslant n$ for all π'. Theorem 5.8 now permits us to read the signature of a representation π' of $O^*(2n)$ off the Langlands parameters, and we get the following:

Corollary 5.22. *Let $\pi' = \pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $O^*(2n)$, and let ε equal the number of zeros in λ (so $\varepsilon = 0$ or 1). Let l be the largest integer with $0 \leqslant l \leqslant r$ such that up to permutations of indices, $\mu = (1, \ldots, 1, \mu_{l+1}, \ldots, \mu_r)$ and $\nu = (1+2\varepsilon, 5+2\varepsilon, \ldots, 4l-3+2\varepsilon, \nu_{l+1}, \ldots, \nu_r)$. Then $sgn(\pi) = (p,q) = (p(\lambda) + r - l, q(\lambda) + r - l)$, and $\theta_{p,q}(\pi) = \pi(r-l, \Gamma\lambda, \Gamma\Psi, \mu', \nu')$, where $\mu' = (\mu_{l+1}, \ldots, \mu_r)$ and $\nu' = (\nu_{l+1}, \ldots, \nu_r)$.*

Starting from the case $p+q = n, n-1$, we can use the one-sided induction principle again to obtain the correspondence for $p+q > n$ in the Witt tower determined by $sgn(\pi)$.

Theorem 5.23. *Let $\pi' = \pi(r, \lambda, \Psi, \mu, \nu) \in \widehat{O^*(2n)}$, $p = p(\lambda) + r$, and $q = q(\lambda) + r$ (so that $p+q = n$ or $n-1$). Let $\pi = \theta_{p,q}(\pi') = \pi(r, \Gamma\lambda, \Gamma\Psi, \mu, \nu) \in \widehat{Sp(p,q)}$ be as in Theorem 5.1. Write $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_r)$. Let s be a non-negative integer. Then*

$$\theta_{p+s,q+s}(\pi') = \pi_s = \pi(r+s, \Gamma\lambda, \Gamma\Psi, \mu^s, \nu_0^s), \tag{5.24}$$

where $\mu^s = (\mu_1, \ldots, \mu_r, \underbrace{1, \ldots, 1}_{s})$, and

$$\nu_0^s = \begin{cases} (\nu_1, \ldots, \nu_r, 3, 7, 11, \ldots, 4s-1) & \text{if } p+q = n, \\ (\nu_1, \ldots, \nu_r, 1, 5, 9, \ldots, 4s-3) & \text{if } p+q = n-1. \end{cases} \tag{5.25}$$

Proof. Very similar to the proof of Theorem 5.8; we omit the details. $\square$

5.3. Applications

Finally, we comment on the applications mentioned near the end of the introduction. By [14], the theta correspondence takes unitary representations of $Sp(p,q)$ to unitary representations of $O^*(2n)$ whenever $p+q \leqslant n/2$. Thus starting from well-known unitary representations of $Sp(p,q)$ (e.g. the trivial representation, tempered representations, unitary representations with non-zero cohomology, etc.), Theorem 5.8 describes the Langlands parameters of the corresponding unitary representations of $O^*(2n)$ which are much less familiar. Indeed, many of them are rather exotic.

Going in the other direction, the theta correspondence takes unitary representations of $O^*(2n)$ to unitary representations of $Sp(p,q)$ whenever $p,q \geqslant n$. From well known unitary representations of $O^*(2n)$ (e.g. the unitary lowest weight modules), Theorem 5.23 describes the Langlands parameters of the corresponding unitary representations of $Sp(p,q)$, for those p,q with $p,q \geqslant n$ and (p,q) in the appropriate Witt tower.

We also know [14] that any irreducible unitary representation of $O^*(2n)$ with rank $r < [n/2]$ must be a theta lift from some unitary representation of $Sp(p,q)$ with $p+q=r$. Now suppose $n \geqslant 8$. Then all irreducible unitary representations of $O^*(2n)$ with rank $\leqslant 3$ are theta lifts from some $Sp(p,q)$ with $p+q \leqslant 3$. Since the unitary dual of any $Sp(p,q)$ with $\min(p,q) \leqslant 1$ is known [3] Theorem 5.8 gives rise to a classification of all irreducible unitary representation of $O^*(2n)$ with rank $\leqslant 3$.

6. The dual pairs $(GL(m,\mathbb{H}), GL(n,\mathbb{H}))$

For the sake of completeness, we describe the correspondence for the dual pairs $(GL(m,\mathbb{H}), GL(n,\mathbb{H}))$. First, we adapt the induction principle of Section 4 to these type II dual pairs; similar calculations may be found in [1]. Recall that $GL(m,\mathbb{H})$ and $GL(n,\mathbb{H})$ are embedded in $Sp(8mn,\mathbb{R})$ as follows (see [6]). Let U_1 and U_2 be right vector spaces of dimension m and n over $\mathbb{H}$, respectively, and let $W = Hom_{\mathbb{H}}(U_1,U_2) \oplus Hom_{\mathbb{H}}(U_2,U_1)$, considered as a real vector space. Define a symplectic form $\langle , \rangle$ on W by $\langle (S_1,T_1),(S_2,T_2)\rangle = tr(S_1T_2 - S_2T_1)$, with $tr(\cdot)$ the reduced trace over $\mathbb{R}$ on $End_{\mathbb{H}}(U_2)$. Then $GL(U_1)\times GL(U_2)$ acts on W by $(g_1,g_2)(S,T) = (g_2Sg_1^{-1}, g_1Tg_2^{-1})$. This action preserves $\langle , \rangle$, and this defines an embedding of $(GL(U_1), GL(U_2)) \cong (GL(m,\mathbb{H}), GL(n,\mathbb{H}))$ as a dual pair into $Sp(W) \cong Sp(8mn,\mathbb{R})$.

For $i=1,2$, let $U_i = V_i \oplus W_i$, direct sums of $\mathbb{H}$ vector spaces, with $dim_{\mathbb{H}}\, V_i = k_i$ and $dim_{\mathbb{H}}\, W_i = l_i$, so that $m = k_1 + l_1$ and $n = k_2 + l_2$. Then W admits the following complete polarization:

$$W = X \oplus Y, \tag{6.1}$$

where

$$X = Hom_{\mathbb{H}}(V_1, U_2) \oplus Hom_{\mathbb{H}}(U_2, W_1)$$

and

$$Y = Hom_{\mathbb{H}}(U_2, V_1) \oplus Hom_{\mathbb{H}}(W_1, U_2).$$

The oscillator representation for the dual pair $(GL(U_1), GL(U_2))$ may be realized on the Schwartz space $\mathscr{S}(Y)$, with the action of the Siegel parabolic $P_S = Stab_{Sp(W)}(X) = M_S N_S$ given by simple formulas (see [19]). For $i = 1, 2$, let $P_i = P(W_i)$ be the stabilizer of W_i in $GL(U_i)$. Then $P_i = M_i N_i$, where $M_i \cong GL(V_i) \times GL(W_i)$, and $N_i \cong Hom_{\mathbb{H}}(V_i, W_i)$. Notice that P_i preserves X hence is contained in P_S. Define the symplectic space W_M with the following complete polarization:

$$W_M = X_M \oplus Y_M, \tag{6.2}$$

where

$$X_M = Hom_{\mathbb{H}}(V_1, V_2) \oplus Hom_{\mathbb{H}}(W_2, W_1)$$

and

$$Y_M = Hom_{\mathbb{H}}(V_2, V_1) \oplus Hom_{\mathbb{H}}(W_1, W_2).$$

Then (M_1, M_2) is a dual pair in $Sp(W_M)$. The associated oscillator representation ω_M may be realized on $\mathscr{S}(Y_M)$. As in Section 4, let $\rho : \mathscr{S}(Y) \to \mathscr{S}(Y_M)$ be the obvious restriction map.

For $i = 1, 2$, let ξ_i be the following character of $M_i = GL(V_i) \times GL(W_i)$:

$$\xi_1(g_1, h_1) = (det(g_1))^{-l_2}(det(h_1))^{k_2},$$

$$\xi_2(g_2, h_2) = (det(g_2))^{-l_1}(det(h_2))^{k_1} \quad \text{for } (g_i, h_i) \in GL(V_i) \times GL(W_i). \tag{6.3}$$

By comparing the actions of the groups involved in the mixed models of ω and ω_M, we have

Proposition 6.4. *The restriction map ρ is a surjective $P_1 \times P_2$-map*

$$\omega \to \omega_M \otimes \xi_1 \otimes \xi_2. \tag{6.5}$$

A calculation as in Section 4 yields the modulus functions ρ_1 and ρ_2 of P_1 and P_2 to be given by

$$\rho_1(g_1, h_1, n_1) = (det(g_1))^{-l_1}(det(h_1))^{k_1},$$

$$\rho_2(g_2, h_2, n_2) = (det(g_2))^{-l_2}(det(h_2))^{k_2} \tag{6.6}$$

for $g_i \in GL(V_i)$, $h_i \in GL(W_i)$, and $n_i \in N_i$. We obtain the induction principle for the dual pair $(GL(U_1), GL(U_2))$.

Theorem 6.7. *For $i = 1, 2$, let $\pi_i \in \widehat{GL(V_i)}$, $\sigma_i \in \widehat{GL(W_i)}$, and suppose that $\pi_1 \leftrightarrow \pi_2$ and $\sigma_1 \leftrightarrow \sigma_2$ in the correspondence for the dual pairs $(GL(V_1), GL(V_2))$ and $(GL(W_1), GL(W_2))$. Let χ_1 and χ_2 be the characters of $GL(V_1) \times GL(V_2)$ and $GL(W_1) \times GL(W_2)$ given by*

$$\begin{aligned} \chi_1(g_1, h_1) &= (det(g_1))^{l_1 - l_2}(det(h_1))^{k_2 - k_1}, \\ \chi_2(g_2, h_2) &= (det(g_2))^{l_2 - l_1}(det(h_2))^{k_1 - k_2} \\ &\quad for\ (g_i, h_i) \in GL(V_i) \times GL(W_i). \end{aligned} \tag{6.8}$$

Then there is a non-zero $GL(U_1) \times GL(U_2)$ equivariant map

$$\Psi : \omega \to Ind_{P_1}^{GL(U_1)}(\pi_1 \otimes \sigma_1 \otimes \chi_1) \otimes Ind_{P_2}^{GL(U_2)}(\pi_2 \otimes \sigma_2 \otimes \chi_2). \tag{6.9}$$

Theorem 6.10. *The extended induction principle* (*Theorem* 4.24) *holds in the setting of Theorem* 6.7 *as well* (*see* [1]).

Now we describe the representations of $G = GL(n, \mathbb{H})$. Let $K \cong Sp(n)$ be a maximal compact subgroup of G. Realize G as

$$U^*(2n) = \left\{ \begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{pmatrix} \middle| X, Y \in \mathfrak{gl}(n, \mathbb{C}) \right\} \cap GL(2n, \mathbb{C}). \tag{6.11}$$

There is only one conjugacy class of theta stable Cartan subalgebras of $\mathfrak{g}_0 = \mathfrak{u}^*(2n)$, whose centralizer is $MA \cong GL(1, \mathbb{H})^n \cong SU(2)^n \times \mathbb{R}^n$. An irreducible representation σ of MA is a tensor product of n irreducible representations of $GL(1, \mathbb{H})$, hence may be specified by an n-tuple of positive integers $\mu = (\mu_1, \ldots, \mu_n)$ and an n-tuple of complex numbers $v = (v_1, \ldots, v_n)$. If $P = MAN$ is a parabolic subgroup of G with Levi factor MA, let $I(\mu, v) = Ind_P^G(\sigma)$. Then $I(\mu, v)$ has a unique lowest K-type constituent $\tau(\mu, v)$ (independent of the choice of P). Two such representations $\tau(\mu, v)$ and $\tau(\mu', v')$ are equivalent if and only if (μ', v') is obtained from (μ, v) by simultaneous permutation of the coordinates, and every irreducible admissible representation of G is of the form $\tau(\mu, v)$ for some choice of μ and v.

The infinitesimal character of $\tau(\mu, v)$ is $(\frac{\mu_1 + v_1}{2}, \ldots, \frac{\mu_n + v_n}{2}, \frac{-\mu_1 + v_1}{2}, \ldots, \frac{-\mu_n + v_n}{2})$, and the unique lowest K-type of $\tau(\mu, v)$ has as its highest weight the dominant Weyl group conjugate of $\mu - 1 = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_n - 1)$. For example, the trivial representation is $\tau(\mu, v)$ with $\mu = (1, \ldots, 1)$ and $v = (2n - 2, 2n - 6, \ldots, -2n + 2)$ and has infinitesimal character $\rho = (\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{-2n+1}{2})$.

Theorem 6.12. *Suppose $m \leqslant n$, and let $d = n - m$. The duality correspondence for the dual pair $(G_1, G_2) = (GL(n, \mathbb{H}), GL(m, \mathbb{H}))$ is given as follows. Let $\tau = \tau(\mu, v)$ be an irreducible admissible representation of $GL(m, \mathbb{H})$. Then τ occurs in the correspondence, and $\theta(\tau) = \tau' = \tau(\mu', v')$, where $\mu' = (\mu_1, \ldots, \mu_m, 1, \ldots, 1)$ and*

$\nu' = (-\nu_1, \dots, -\nu_m, 2d-2, 2d-6, \dots, -2d+2)$. *In particular, the correspondence for the dual pairs* $(GL(n,\mathbb{H}), GL(n,\mathbb{H}))$ *is given by* $\tau \leftrightarrow \tau^*$, *with all representations of* $GL(n,\mathbb{H})$ *occurring. Moreover, the lowest K-types of* τ *and* τ' *are of minimal degree and correspond in the space of joint harmonics.*

The proof of Theorem 6.12 proceeds along the same lines as that of Theorem 5.1. We first prove two facts which will help us apply Theorem 6.10.

Lemma 6.13. *Let* τ *be an irreducible admissible representation of* $GL(n,\mathbb{H})$, *with lowest K-type* $\eta = (a_1, \dots, a_n)$. *Then* η *is of minimal degree in* τ.

Proof. By Frobenius Reciprocity, every $Sp(n)$-type of τ contains the weight $(a_1, \dots, a_n)$, hence its highest weight is of the form $(a_1, \dots, a_n) +$ a sum of positive roots. Since the degree of any $Sp(n)$-type is given by the sum of the coefficients of its highest weight, and the positive roots are those of the form $2e_i$ and $e_i \pm e_j$, the result follows easily. □

Lemma 6.14. *Let* $n = k + l$, *and let* $\pi = \tau(\mu, \nu)$ *and* $\pi' = \tau(\mu', \nu')$ *be irreducible admissible representations of* $GL(k,\mathbb{H})$ *and* $GL(l,\mathbb{H})$, *respectively. Let P be a maximal parabolic subgroup of* $GL(n,\mathbb{H})$ *with Levi factor* $M \cong GL(k,\mathbb{H}) \times GL(l,\mathbb{H})$, *and let* $I = Ind_P^{GL(n,\mathbb{H})}(\pi \otimes \pi')$. *Then I has a unique lowest K-type* δ. *Let* η *and* η' *be the unique lowest K-types of* π *and* π', *respectively. Then the restriction of* δ *to* $Sp(k) \times Sp(l)$ *contains* $\eta \otimes \eta'$, *and the degree of* δ *equals the degree of* $\eta \otimes \eta'$.

Proof. Using induction by stages, we know that I contains (as its lowest K-type constituent) the representation $\sigma = \tau(\mu^0, \nu^0)$ of $GL(n,\mathbb{H})$, where μ^0 is obtained from μ and μ' by concatenation, and similarly for ν^0, ν, and ν'. Now σ has a unique lowest K-type whose highest weight (Weyl group conjugate to) $\mu^0 - (1, \dots, 1)$ is also the highest weight of $\eta \otimes \eta'$. So the restriction of δ to $Sp(k) \times Sp(l)$ contains $\eta \otimes \eta'$, and the statement of the degrees follows from Proposition 4.25. □

Proof of Theorem 6.12. We start with the case $m = n$, using induction on m, with the base case $m = 1$ given by Proposition 4.27. If $m \geqslant 2$, we let $k_1 = k_2 = m - 1$ and $l_1 = l_2 = 1$ to get P_1, P_2, and the map ρ of Proposition 6.4. Let $\tau = \tau(\mu, \nu)$ be an irreducible admissible representation of $GL(m,\mathbb{H}) = G_1$. If P is a minimal parabolic subgroup of G_1 (with Levi factor $M \cong GL(1,\mathbb{H})^m$) which is contained in P_1, we can rearrange the indices of μ and ν so that $Re\,\nu$ is dominant with respect to P. Then τ is the unique irreducible quotient of $Ind_P^{G_1}(\sigma(\mu_1, \nu_1) \otimes \cdots \otimes \sigma(\mu_m, \nu_m))$, and, by Frobenius reciprocity, of $Ind_{P_1}^{G_1}(\tau_1 \otimes \sigma(\mu_m, \nu_m))$, where $\tau_1 = \tau(\mu^0, \nu^0) \in \widehat{GL(m-1,\mathbb{H})}$, and μ^0 and ν^0 are obtained from μ and ν by removing the last coordinate. By the inductive hypothesis and Theorem 6.7, we have a non-zero map

$$\Psi : \omega \rightarrow Ind_{P_1}^{G_1}(\tau_1 \otimes \sigma(\mu_m, v_m)) \otimes Ind_{P_2}^{G_2}(\tau_1^* \otimes \sigma(\mu_m, -v_m)) \tag{6.15}$$

(taking $\pi \otimes \sigma = \tau_1 \otimes \sigma(\mu_m, v_m) \otimes \chi_1^*$). The result now follows using Theorem 6.10, Proposition 4.25, Lemmas 6.13 and 6.14, analogously to the proof of Theorem 5.1.

For the case $n > m$ we apply Theorem 6.7 to the case $k_1 = k_2 = m$, $l_1 = 0$, $l_2 = d$, and $\pi = \tau \otimes \chi_1^*$ to get a nonzero map

$$\Psi : \omega \rightarrow \tau \otimes Ind_{P_2}^{G_2}(\tau^* \otimes \mathbb{1}). \tag{6.16}$$

The representation τ' is the unique lowest K-type constituent of the induced representation in 6.16. The theorem now follows from the fact that the lowest K-types of τ and τ' correspond in the space of joint harmonics (see Proposition 4.25). $\square$

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